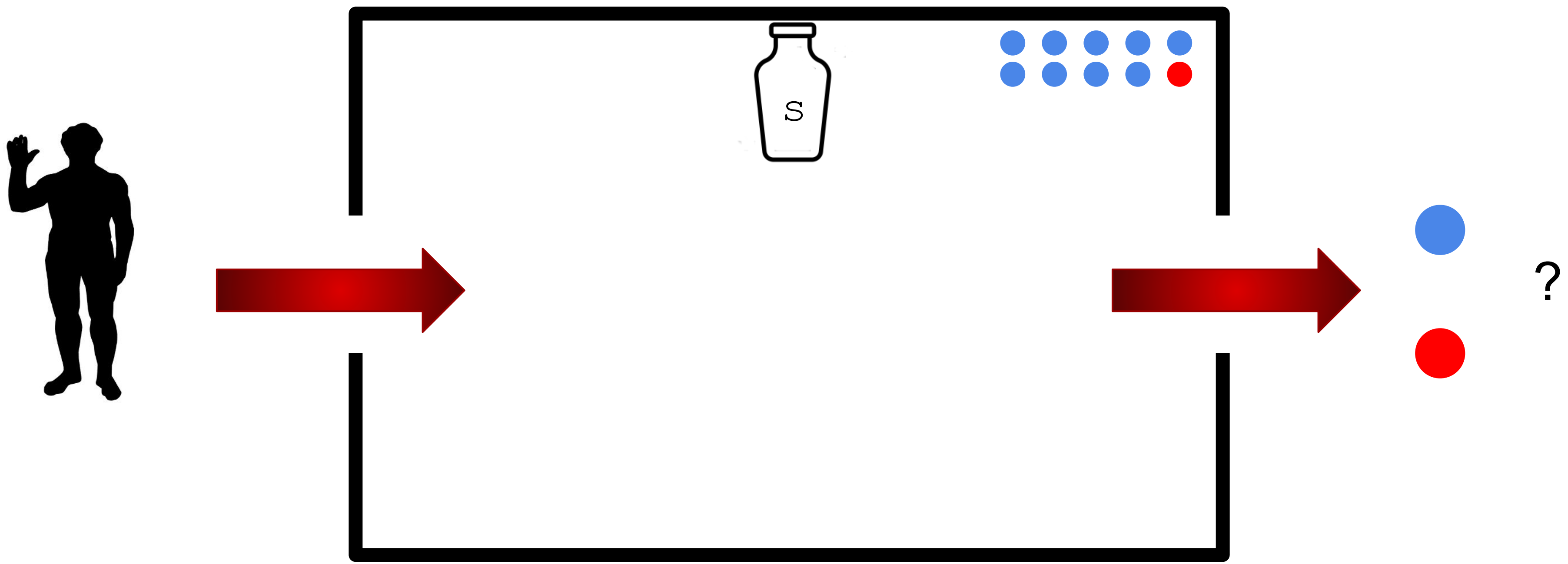
DTU

DTU Health Technology
Bioinformatics

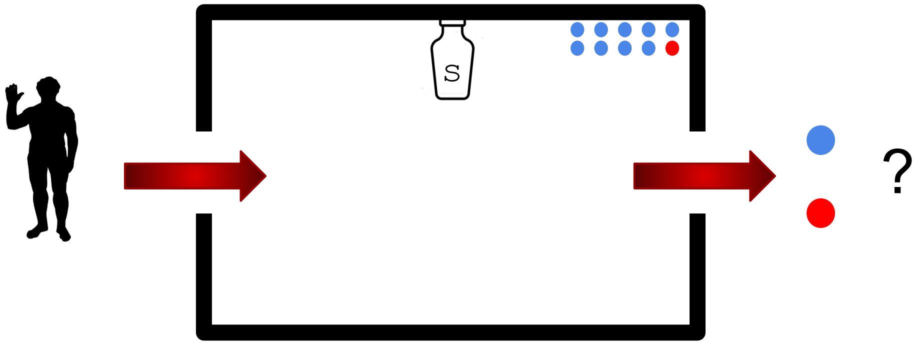
Brief refresher on conditional probabilities and the Bayesian theorem

Gabriel Renaud
Associate Professor
Section of Bioinformatics
Technical University of Denmark
gabriel.reno@gmail.com

Brief probability reminder … but first a little game!



Brief probability reminder

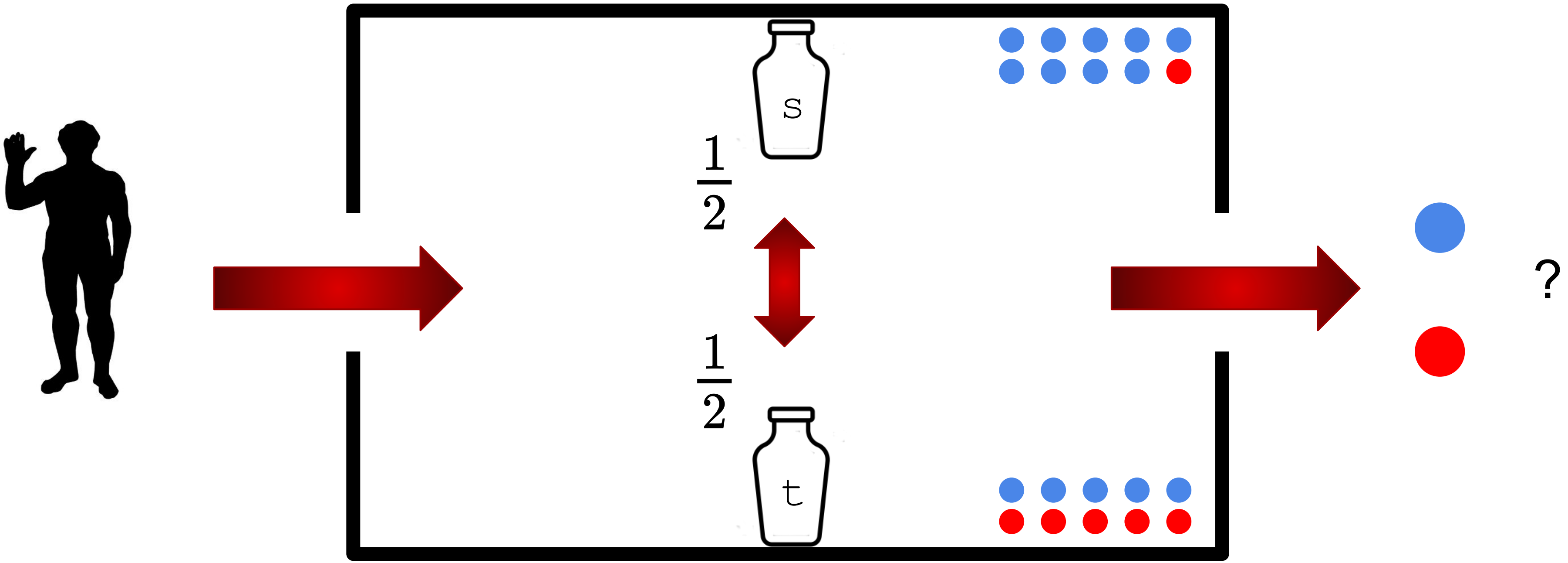


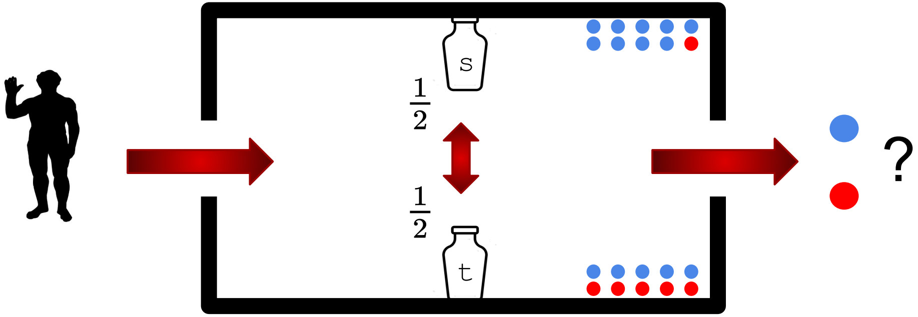
Events:

E = our player picked a red ball

$$P(E) = \frac{1}{10} = 0.1$$

Brief probability reminder





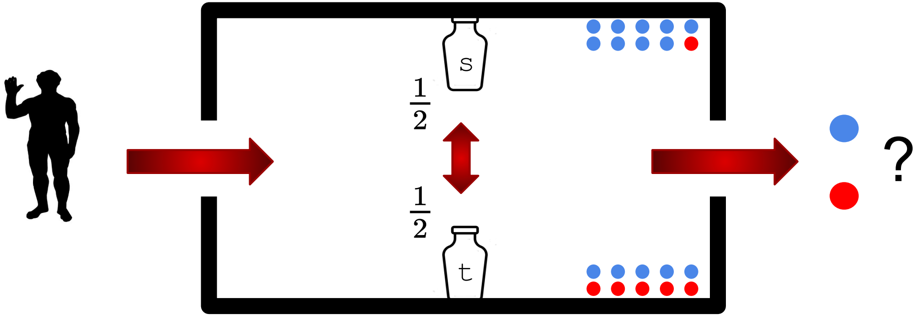
E = our player picked a red ball

S = our player picked the ‘s’ urn

$$P(S) = \frac{1}{2}$$

$$P(E|S) = \frac{1}{10} = 0.1$$

conditional probability (assuming our player picked the ‘s’ urn)



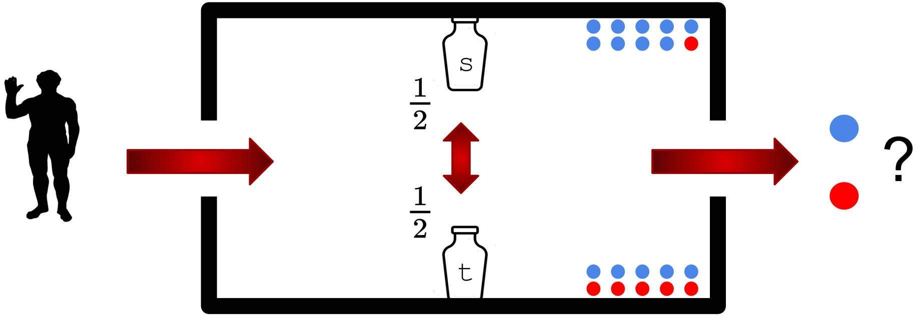
E = our player picked a red ball

S = our player picked the 's' urn

T = our player picked the 't' urn

$$P(T) = \frac{1}{2}$$

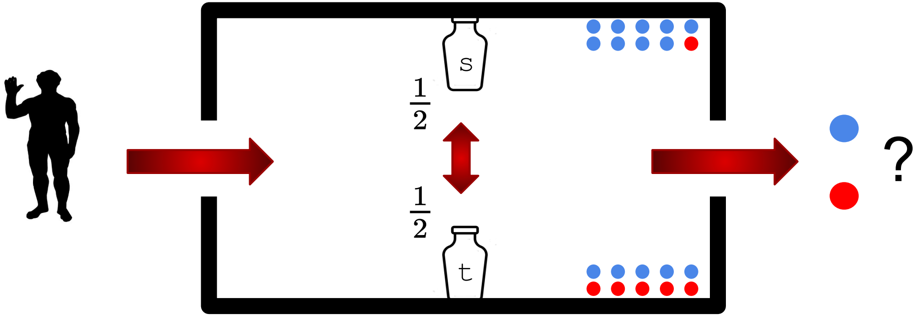
$$P(E|T) = \frac{5}{10} = \frac{1}{2} = 0.5$$



$$S \qquad\qquad T$$

$P(E) =$ (Our player picked urn ‘s’ and picked a red ball) + (Our player picked urn ‘t’ and picked a red ball)

$$P(E) = P(S)P(E|S) + P(T)P(E|T)$$

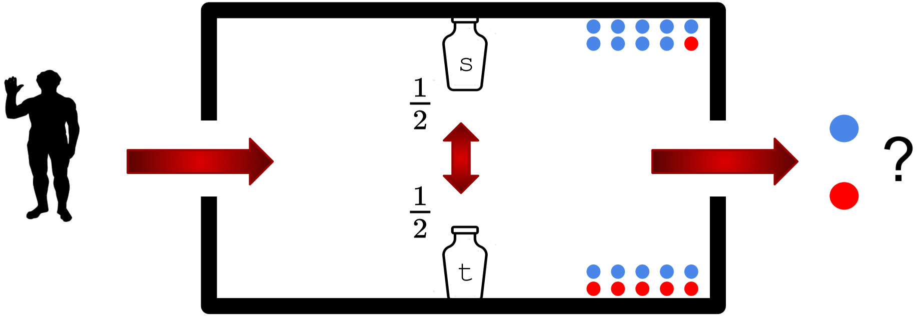


$$P(E) = P(S)P(E|S) + P(T)P(E|T)$$

$$P(E) = \frac{1}{2} \quad \frac{1}{10} + \frac{1}{2} \quad \frac{5}{10}$$

$$P(E) = \frac{1}{20} + \frac{5}{20}$$

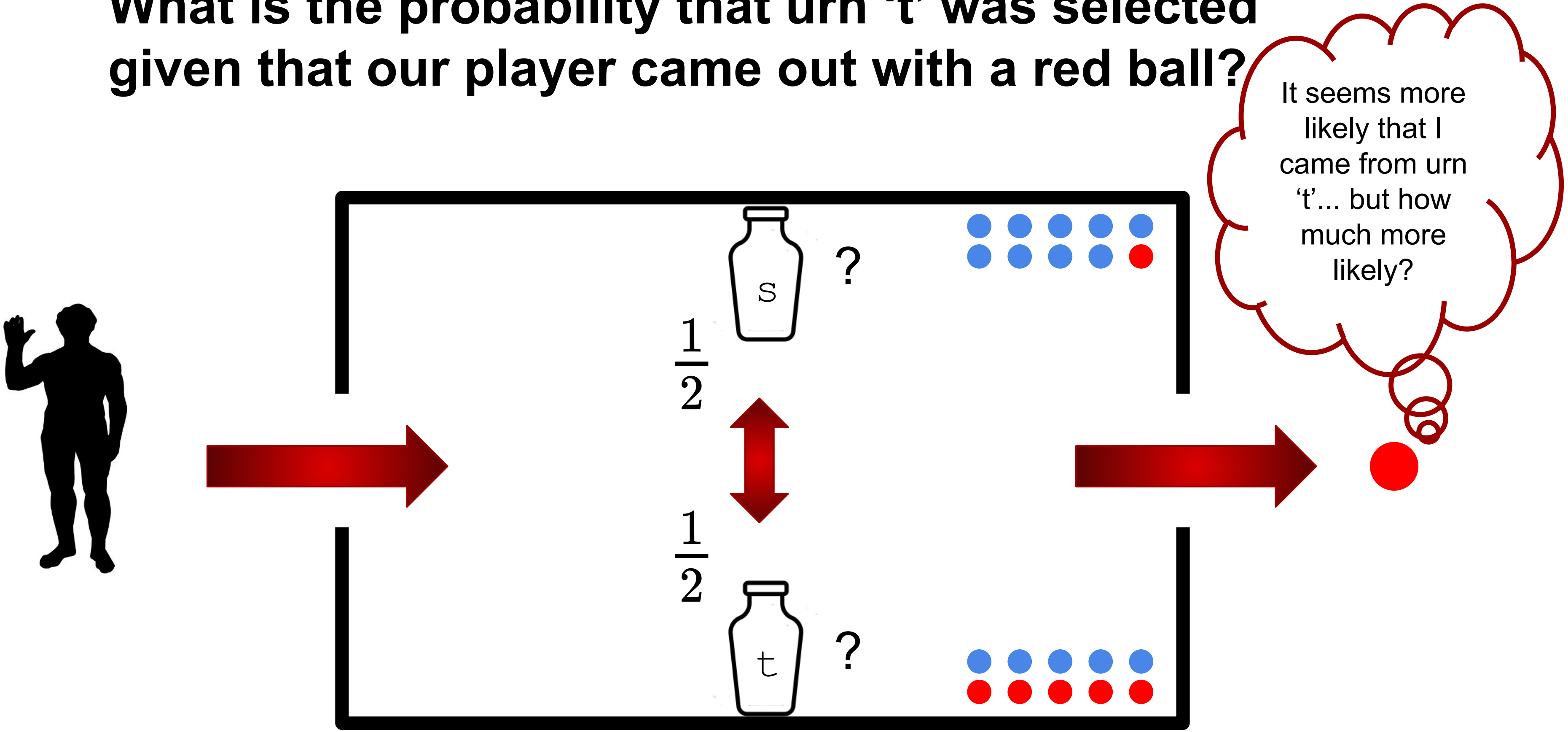
$$P(E) = \frac{6}{20}$$

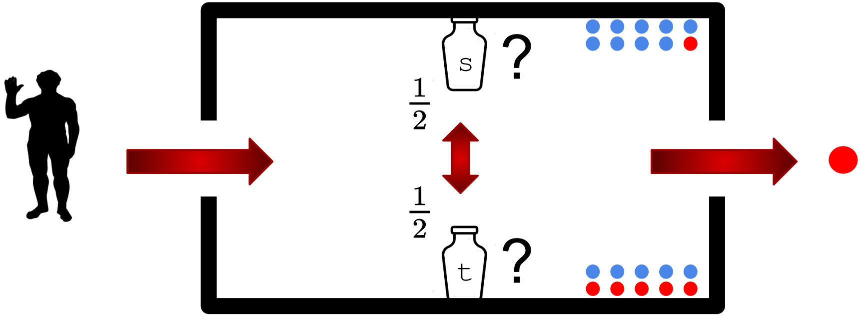


$$P(E) = \frac{6}{20}$$

There is a 30% chance of getting a red ball

What is the probability that urn ‘t’ was selected given that our player came out with a red ball?



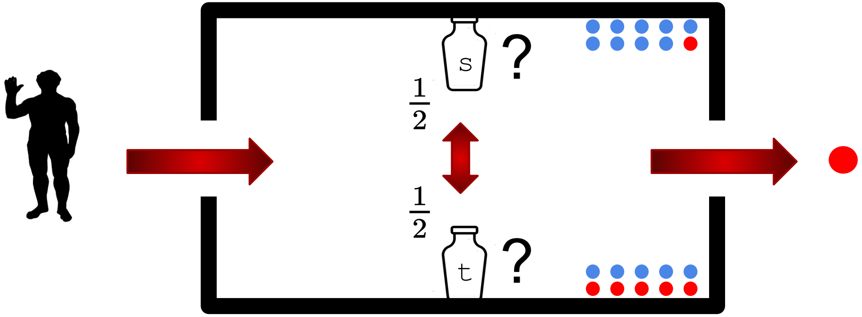


E = our player picked a red ball

T = our player picked the 't' urn

We seek:

$$P(T|E)$$



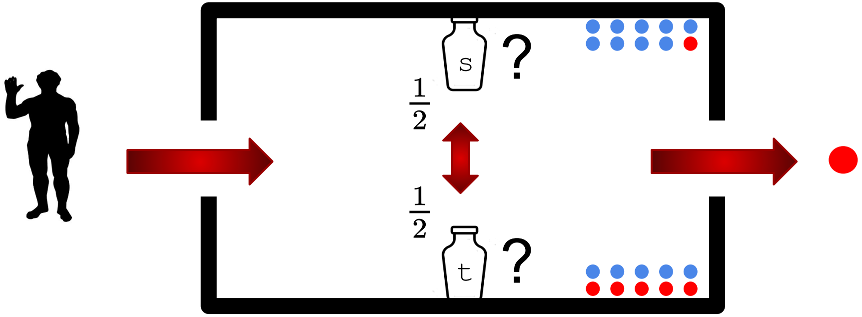
E = our player picked a red ball

T = our player picked the 't' urn

Bayes' Theorem

Thomas Bayes (1701 - 1761)

$$P(T|E) = \frac{P(T)P(E|T)}{P(E)}$$

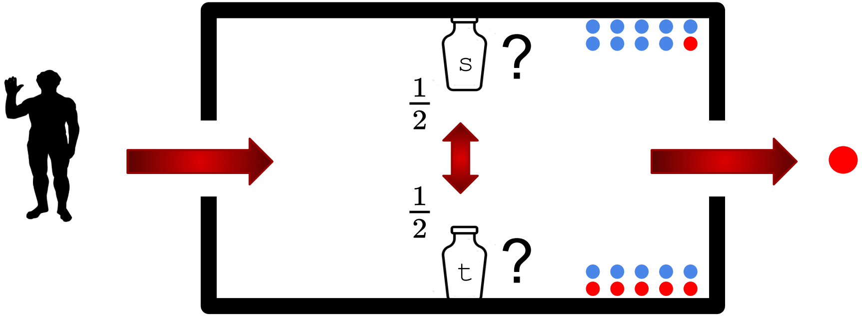


E = our player picked a red ball

T = our player picked the ‘t’ urn

What is the **prior** probability of picking urn ‘t’?

$$P(T|E) = \frac{P(T)P(E|T)}{P(E)}$$

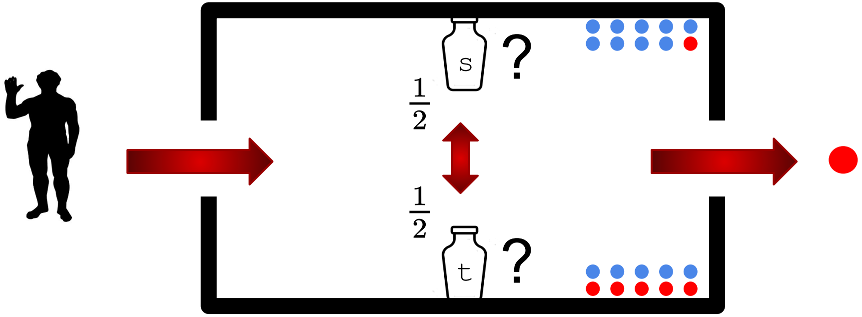


E = our player picked a red ball

T = our player picked the ‘t’ urn

What is the **prior** probability of selecting urn ‘t’?

$$P(T|E) = \frac{\frac{1}{2} \quad P(E|T)}{P(E)}$$

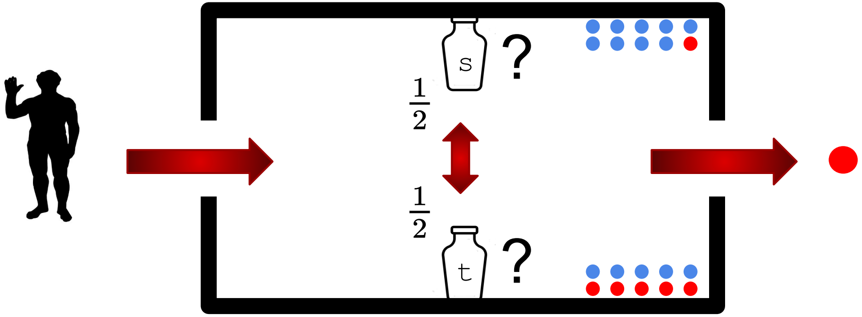


E = our player picked a red ball

T = our player picked the ‘t’ urn

What is the probability of sampling a red ball given than I selected the urn ‘t’?

$$P(T|E) = \frac{\frac{1}{2} \quad P(E|T)}{P(E)}$$

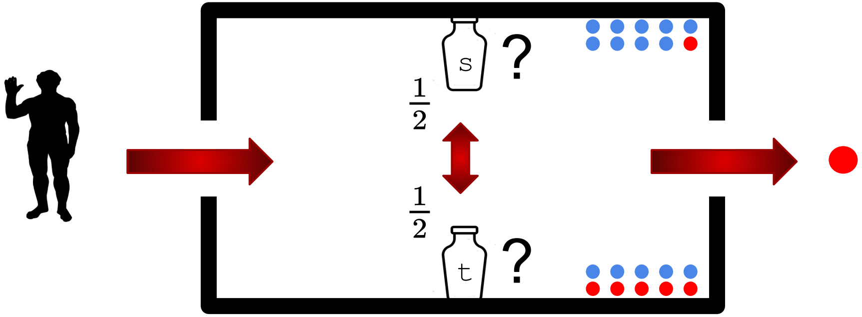


E = our player picked a red ball

T = our player picked the ‘t’ urn

What is the probability of sampling a red ball given than I selected the urn ‘t’?

$$P(T|E) = \frac{\frac{1}{2} \quad \frac{5}{10}}{P(E)}$$

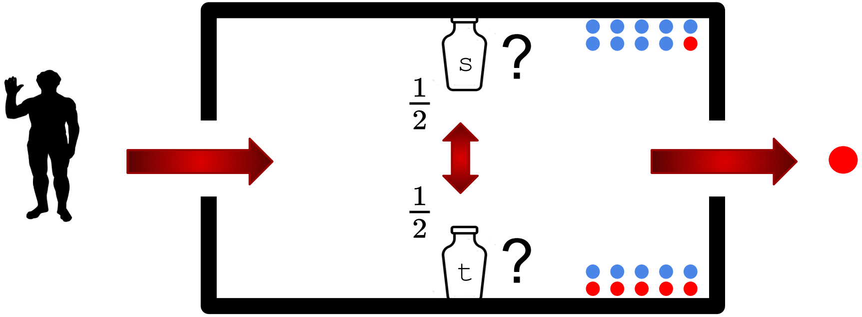


E = our player picked a red ball

T = our player picked the ‘t’ urn

$$P(T|E) = \frac{\frac{1}{2} \quad \frac{5}{10}}{P(E)}$$

What is the probability of sampling a red ball altogether?

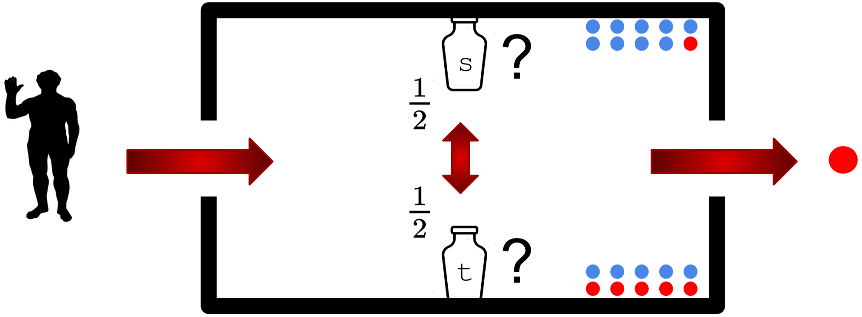


E = our player picked a red ball

T = our player picked the ‘t’ urn

$$P(T|E) = \frac{\frac{1}{2} \qquad \frac{5}{10}}{\frac{6}{20}}$$

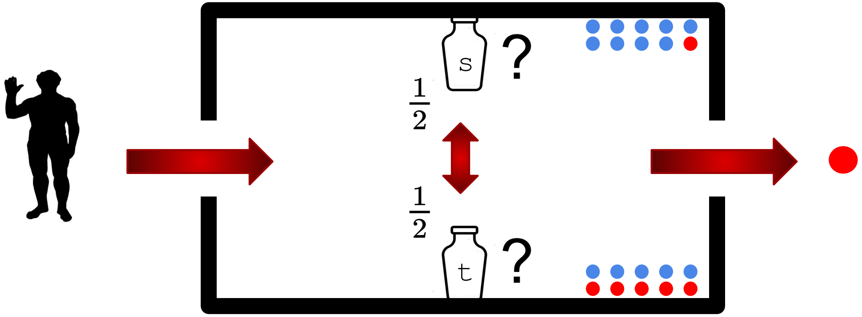
What is the probability of sampling a red ball altogether?



E = our player picked a red ball

T = our player picked the ‘t’ urn

$$P(T|E) = \frac{\frac{5}{20}}{\frac{6}{20}}$$



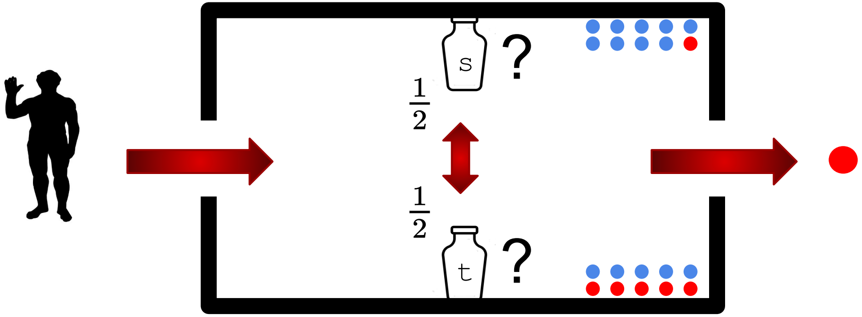
E = our player picked a red ball

T = our player picked the 't' urn

Another way to visualize:

RED	RED	RED	RED	RED
RED	BLUE	BLUE	BLUE	BLUE
BLUE	BLUE	BLUE	BLUE	BLUE
BLUE	BLUE	BLUE	BLUE	BLUE

$$P(T|E) = \frac{\frac{5}{20}}{\frac{6}{20}}$$



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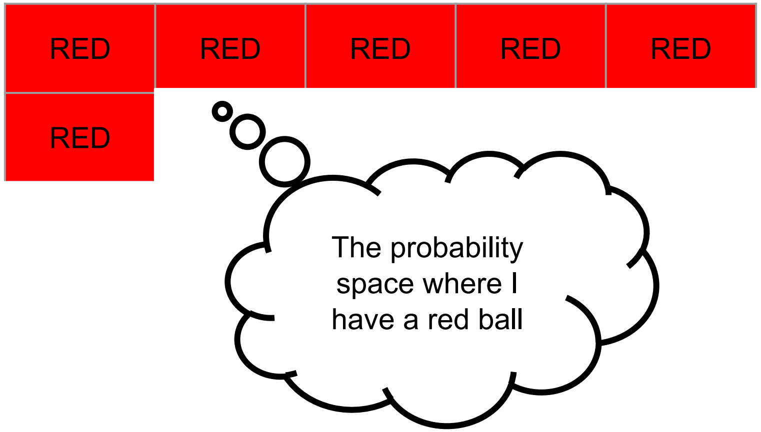
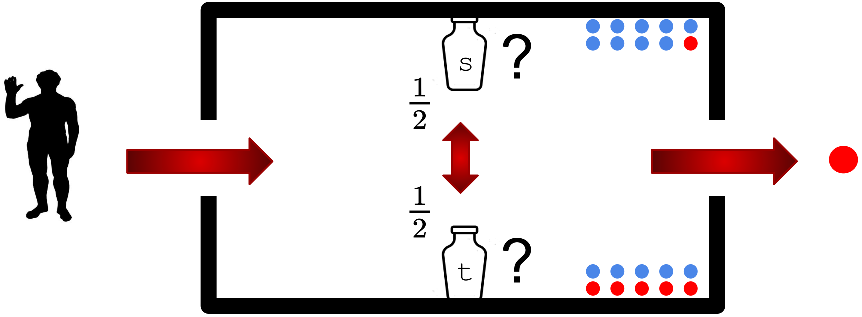
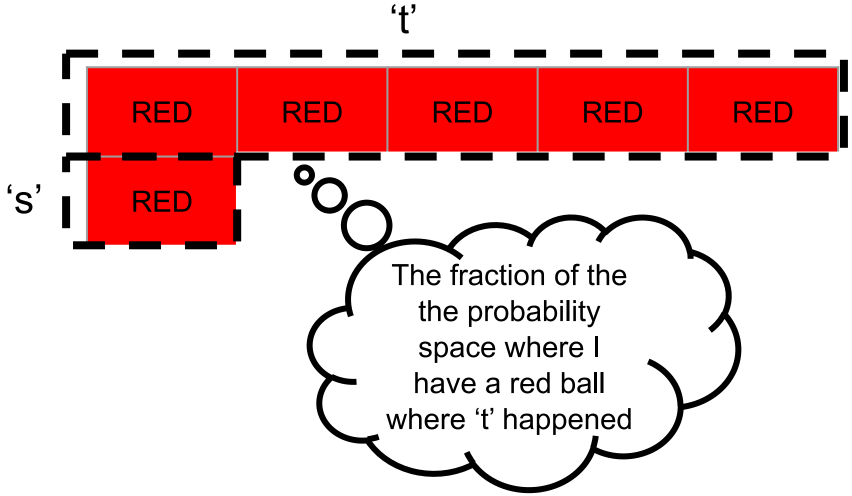
Another way to visualize:



$$P(T|E) = \frac{\frac{5}{20}}{\frac{6}{20}}$$

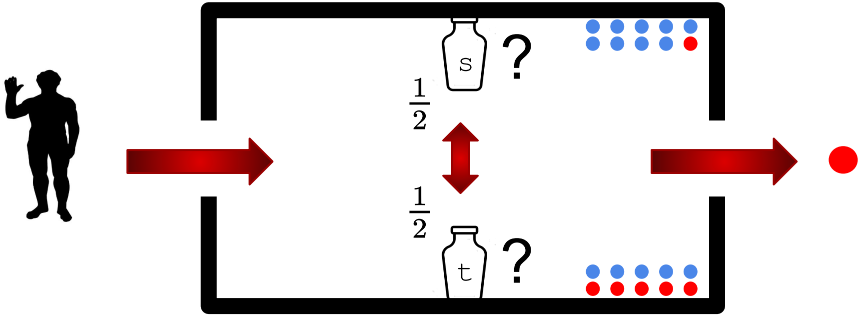




E = our player picked a red ball

T = our player picked the ‘t’ urn

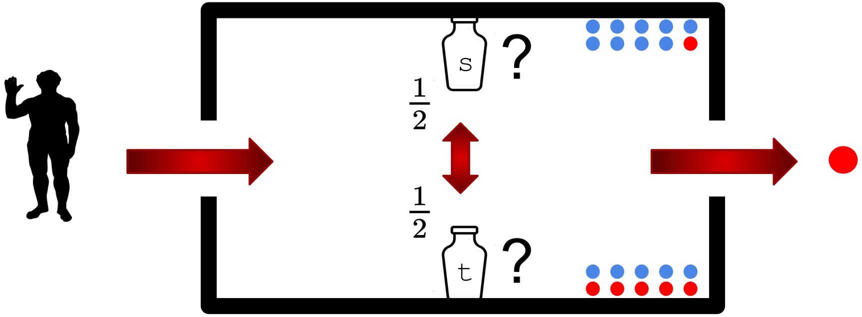
$$P(T|E) = \frac{\frac{5}{20}}{\frac{6}{20}}$$



E = our player picked a red ball

T = our player picked the 't' urn

$$P(T|E) = \frac{5}{6} \approx 83\%$$



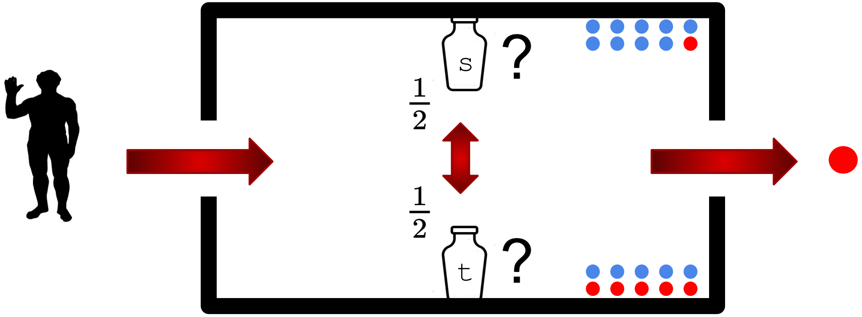
Let us think about Bayes' theorem a bit more...

E = our player picked a red ball

T = our player picked the 't' urn

- The color of the ball is an observation
- The urn that was selected is a piece of information I cannot have access to, a mental model
- I made a prediction about the probability of a model being correct given an observation

$$P(T|E) = \frac{P(T)P(E|T)}{P(E)}$$



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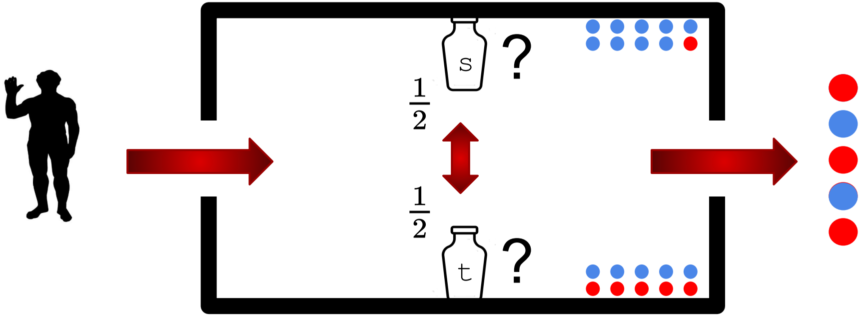
- The color of the ball is an observation
- The urn that was selected is a piece of information I cannot have access to, a mental model
- I made a prediction about the probability of a model being correct given an observation

M = a model

D = our data, our observation

$$P(M|D) = \frac{P(M)P(D|M)}{P(D)}$$



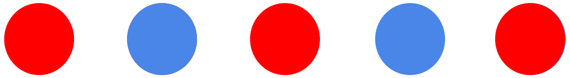
M = a model

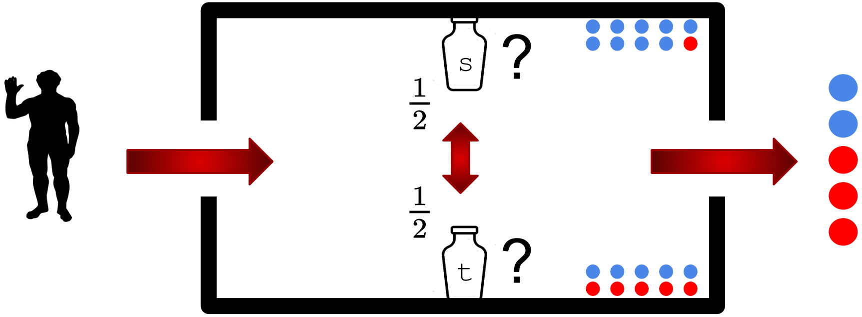
D = our data, our observation

Let us think about Bayes' theorem a bit more...

- Say our player:
  - selects an urn at random
  - picks a ball
  - records it
  - picks a ball again the **same** urn
- Our player does this 5 times
- When he leaves, he reports his observations

Observations 1:





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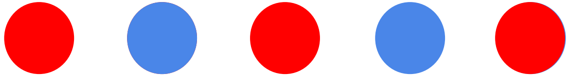
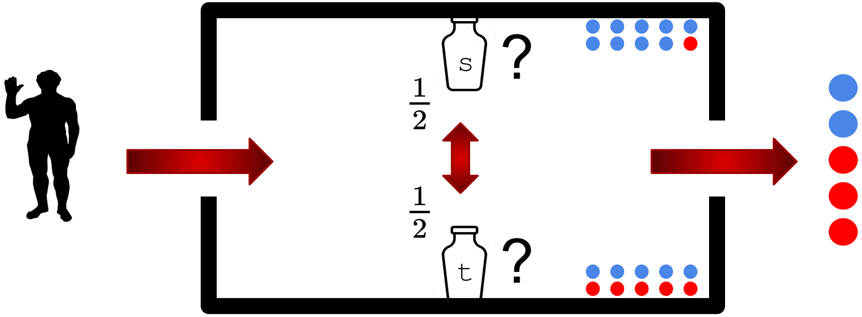
Observations 1:



What is the probability that urn 't' was selected? ~97%



M = a model

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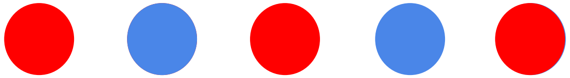
Observations 1:



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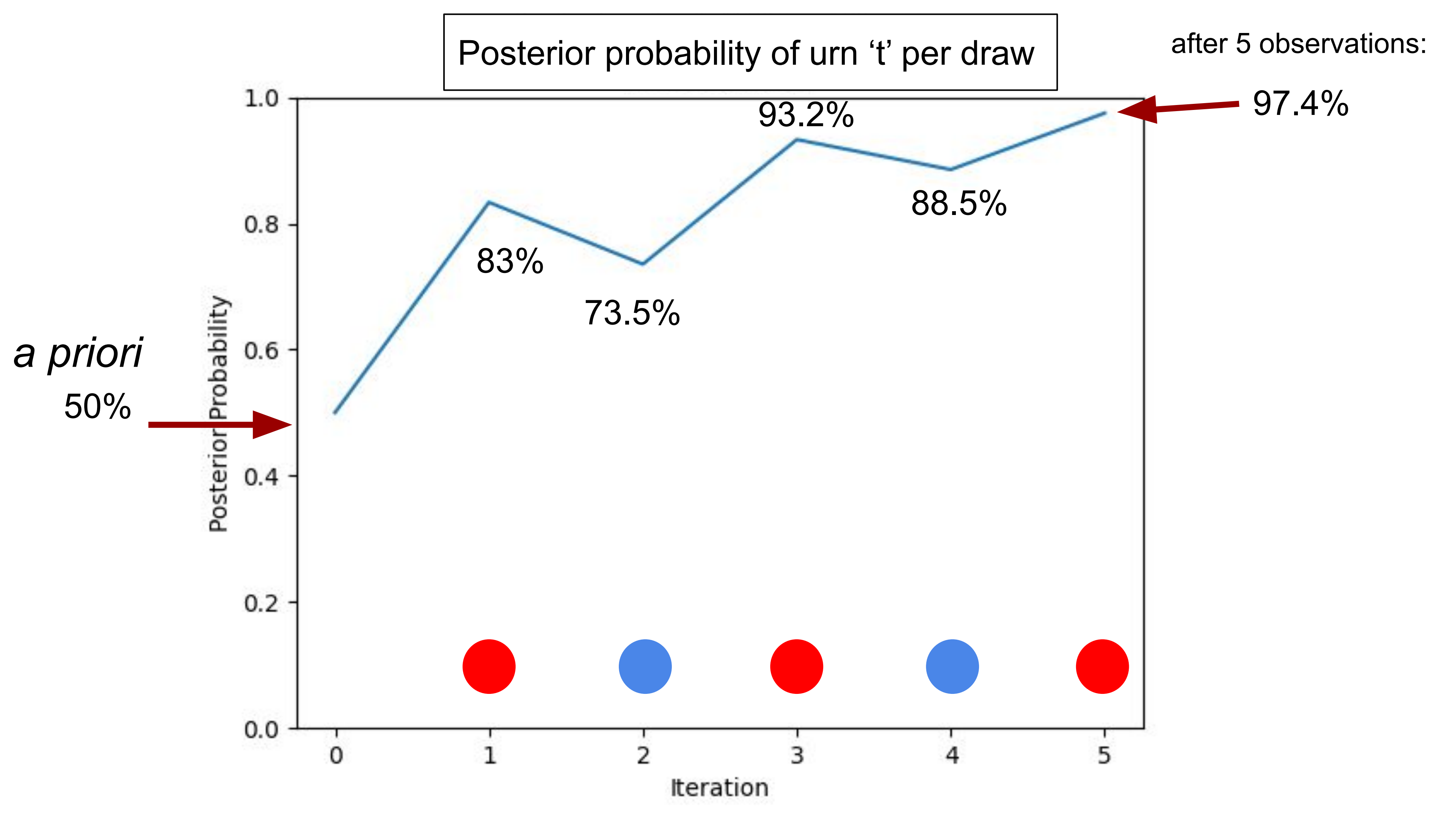
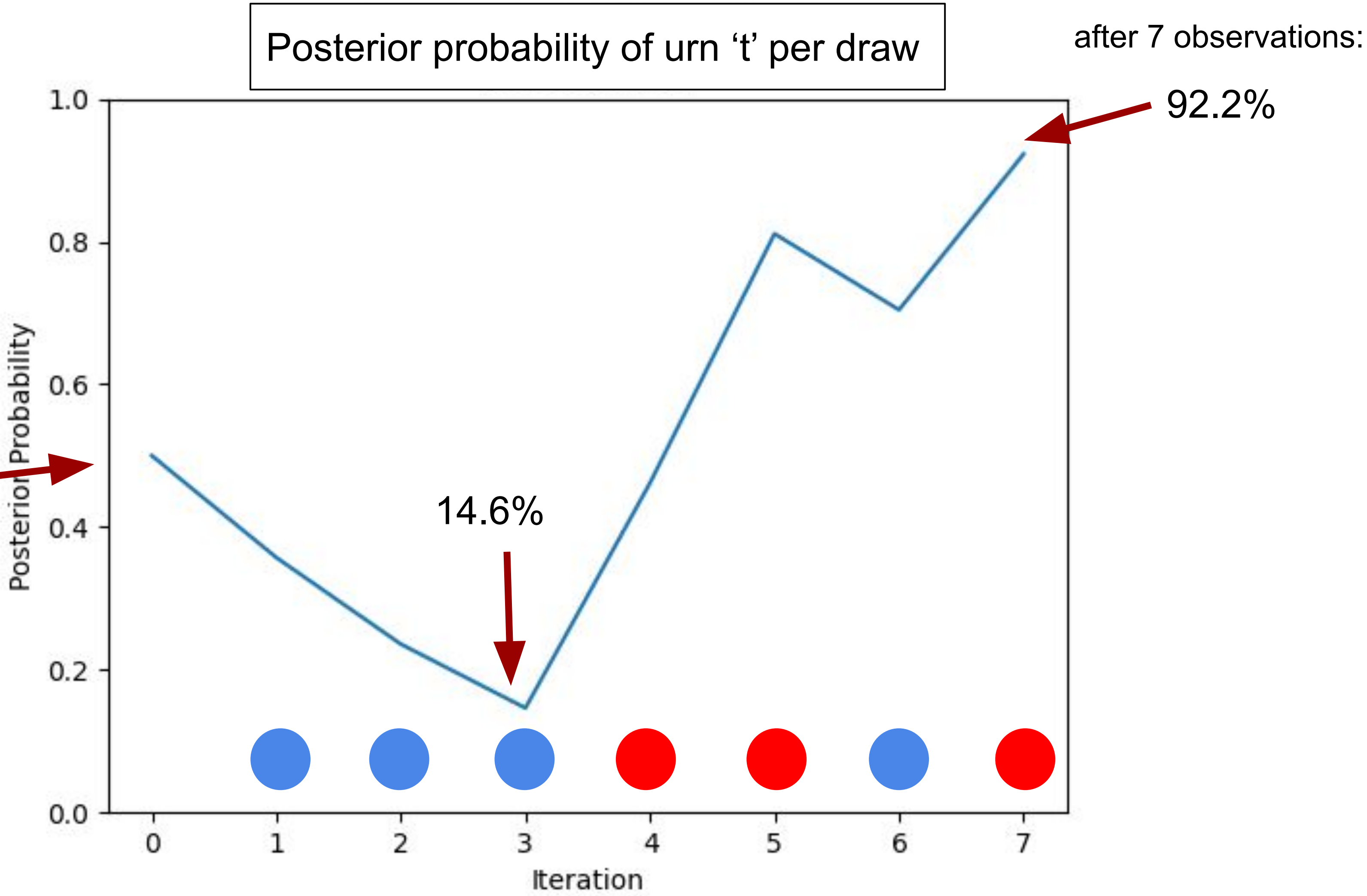
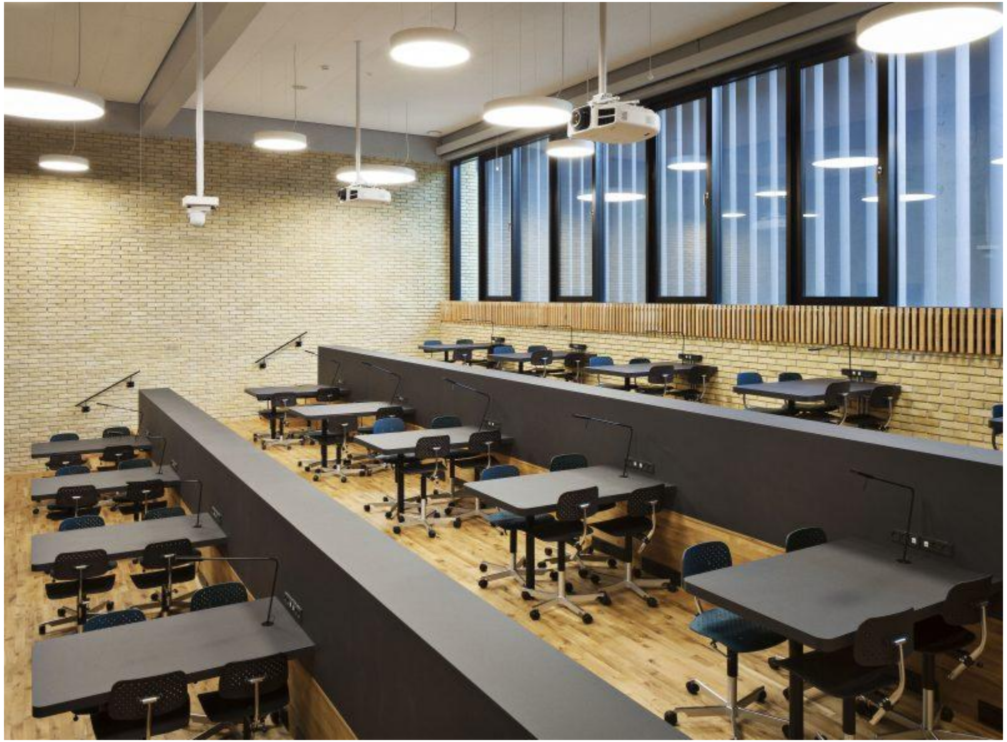

Posterior probability of urn ‘t’ per draw
after 5 observations:
97.4%
93.2%
88.5%
83%
73.5%
a priori
50%
Posterior Probability
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
Iteration

Bayes can "change its mind" if new data becomes available

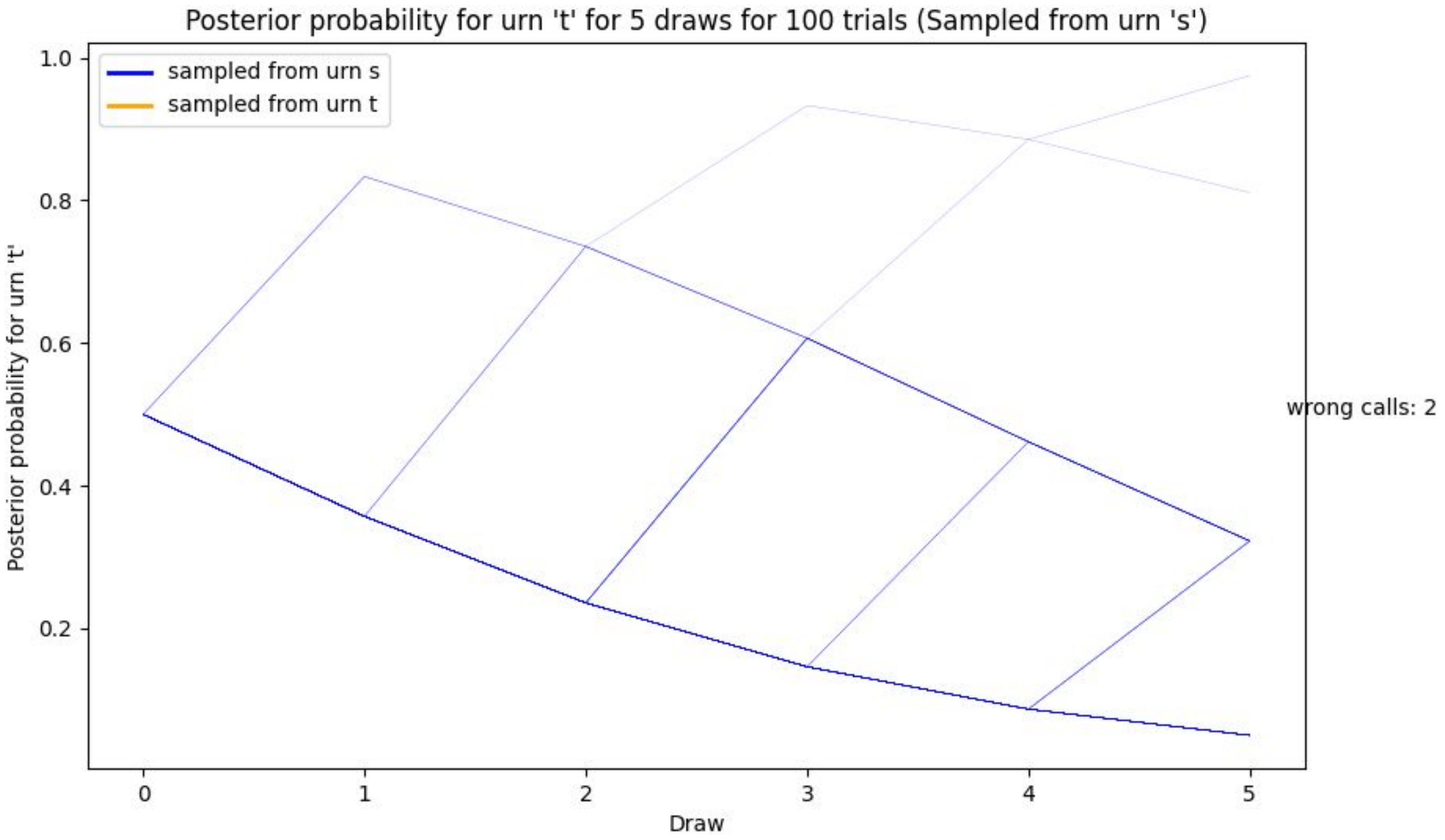


Let’s get 100 people to repeat this experiment (picking 5 balls) and see if we can predict which urn they picked.

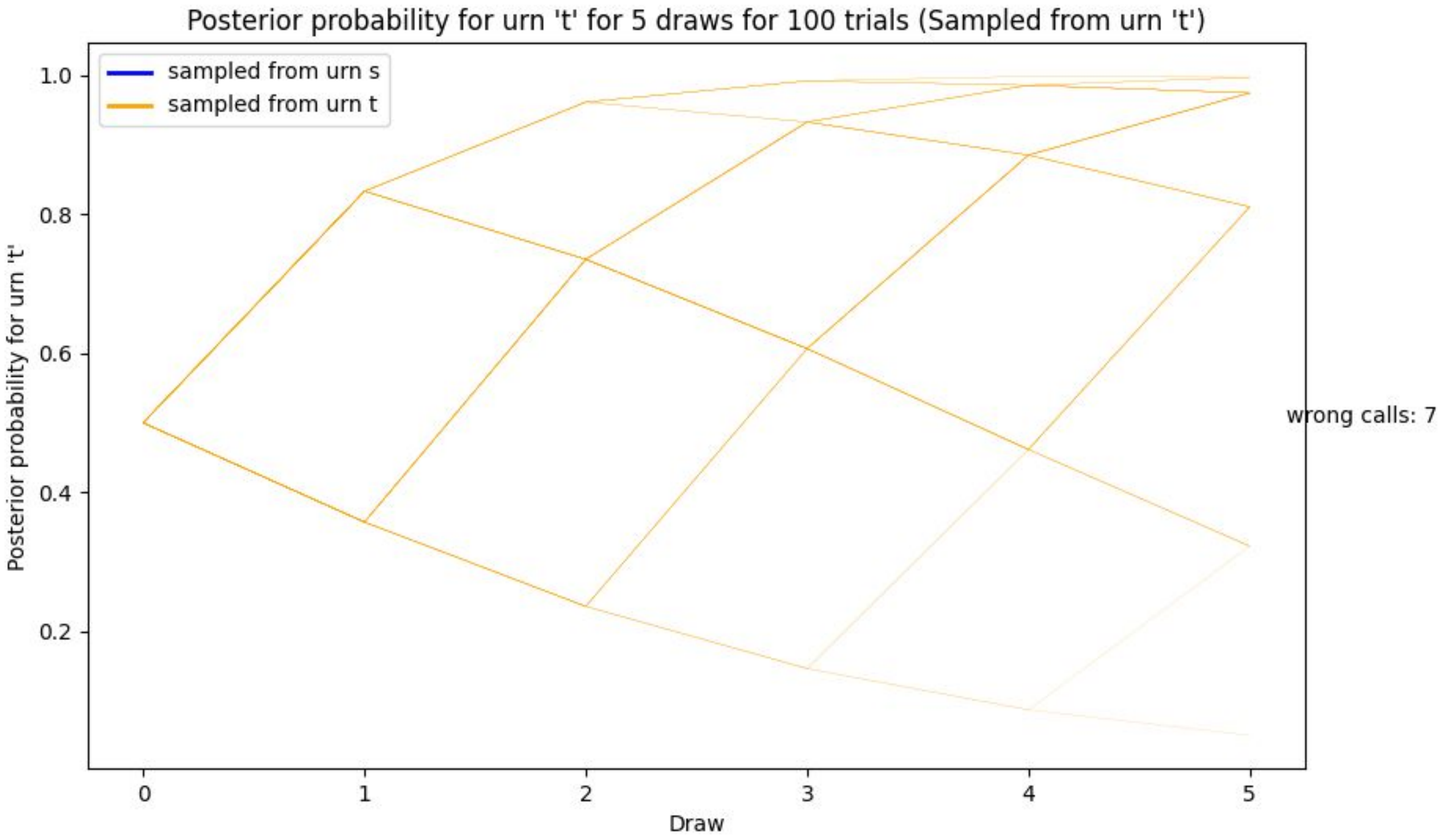


source: renover.dk

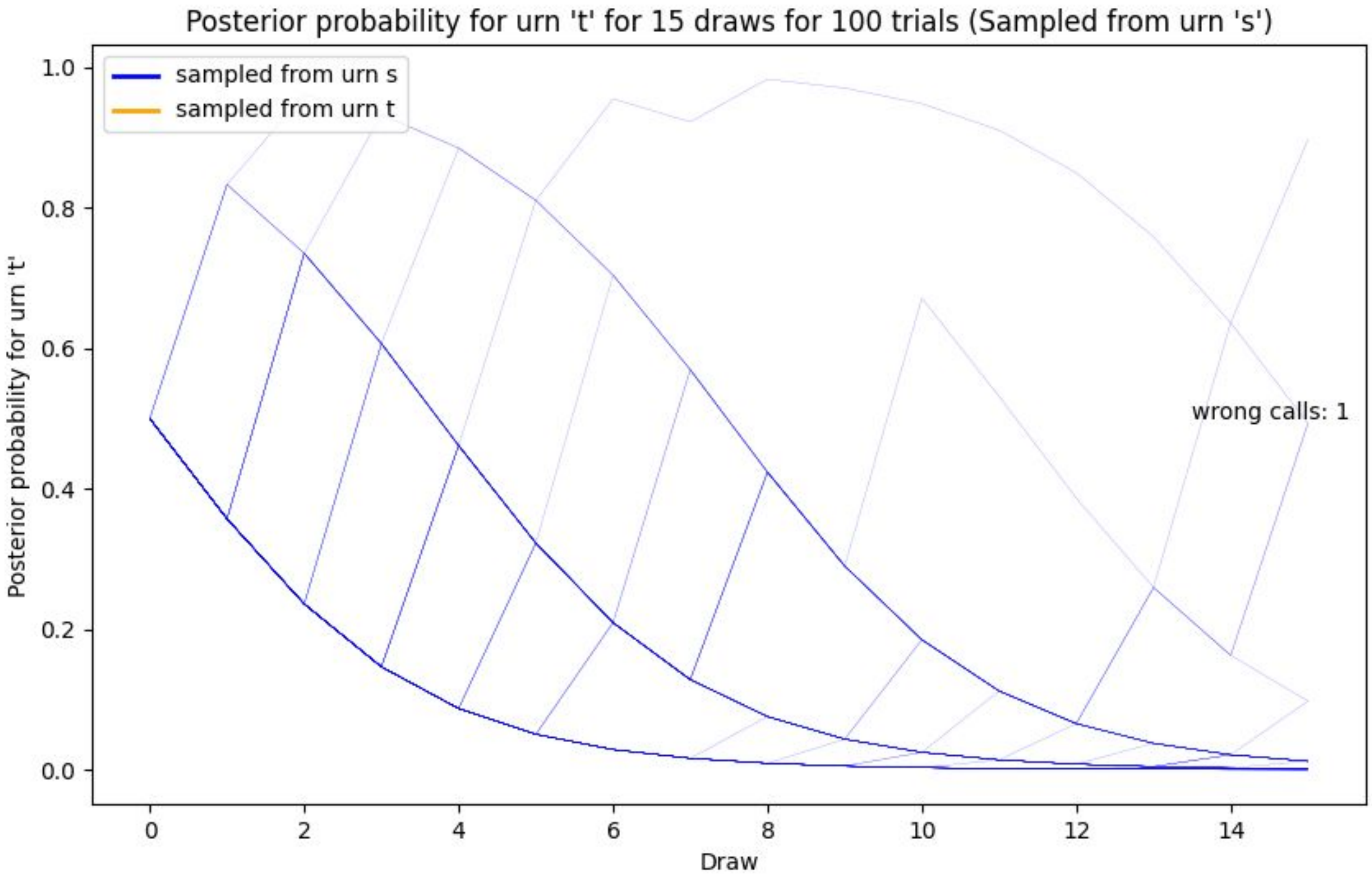
If 100 people picked urn ‘s’



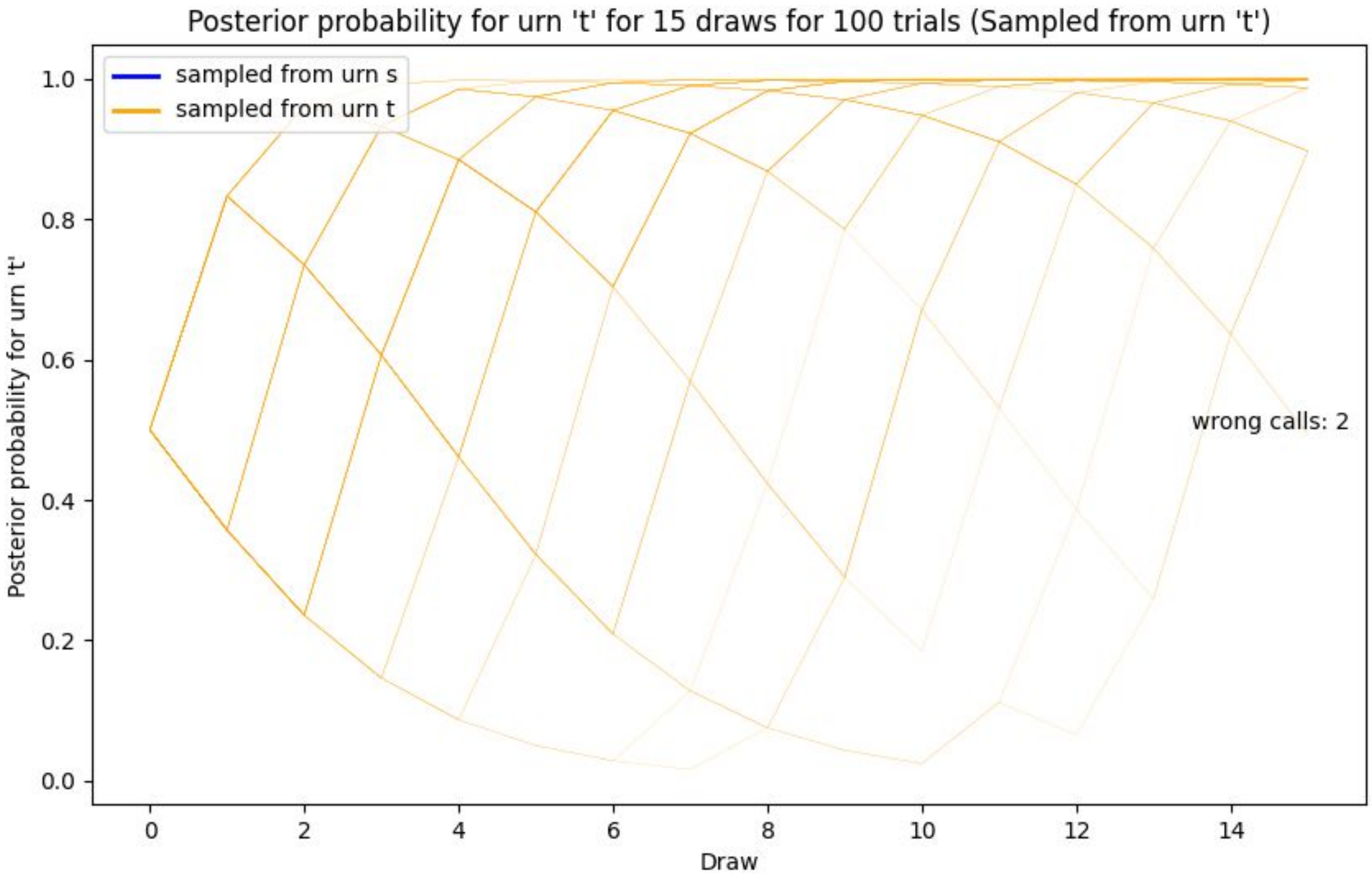
If 100 people picked urn ‘t’

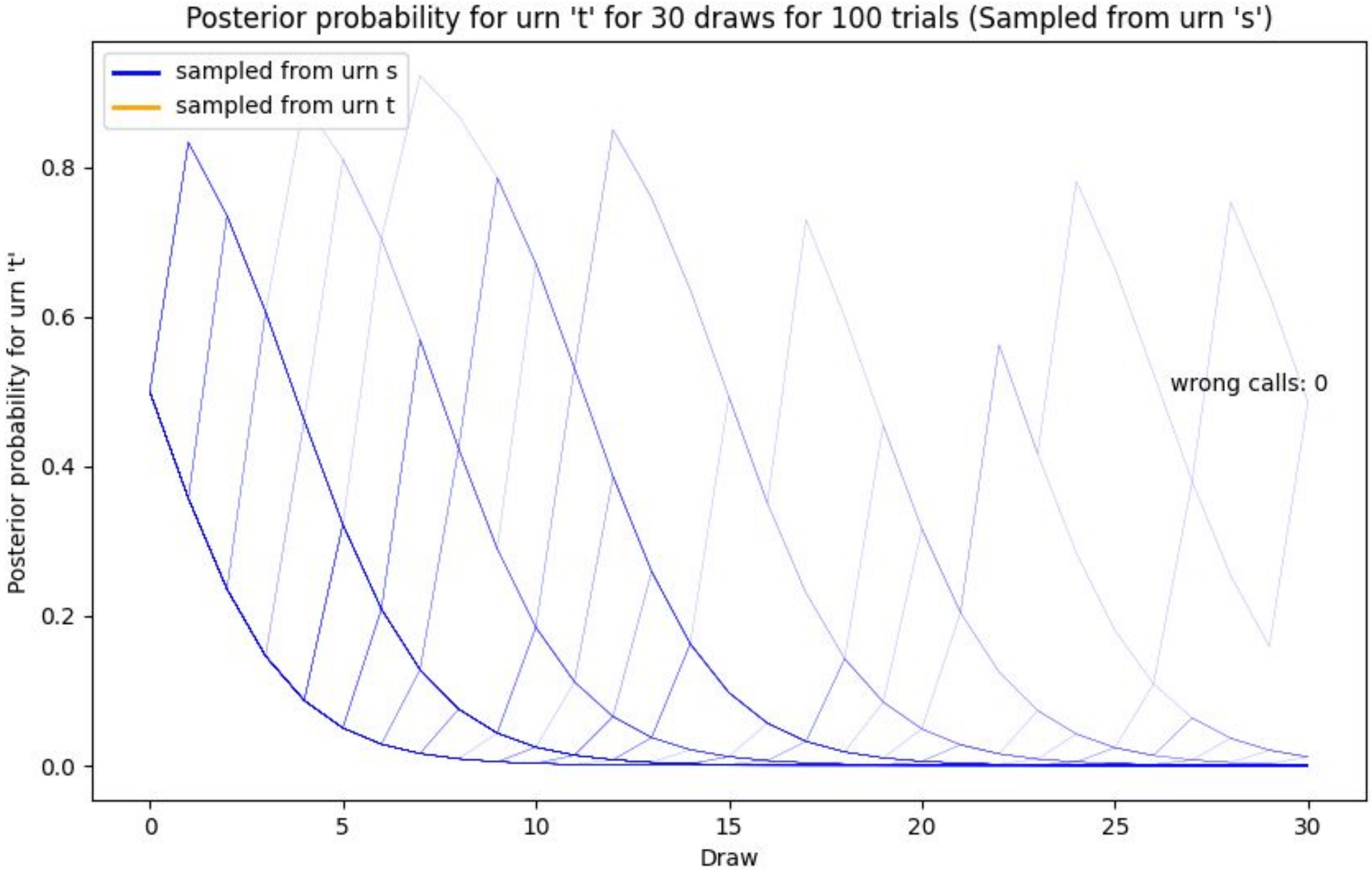


If 100 people picked urn ‘s’

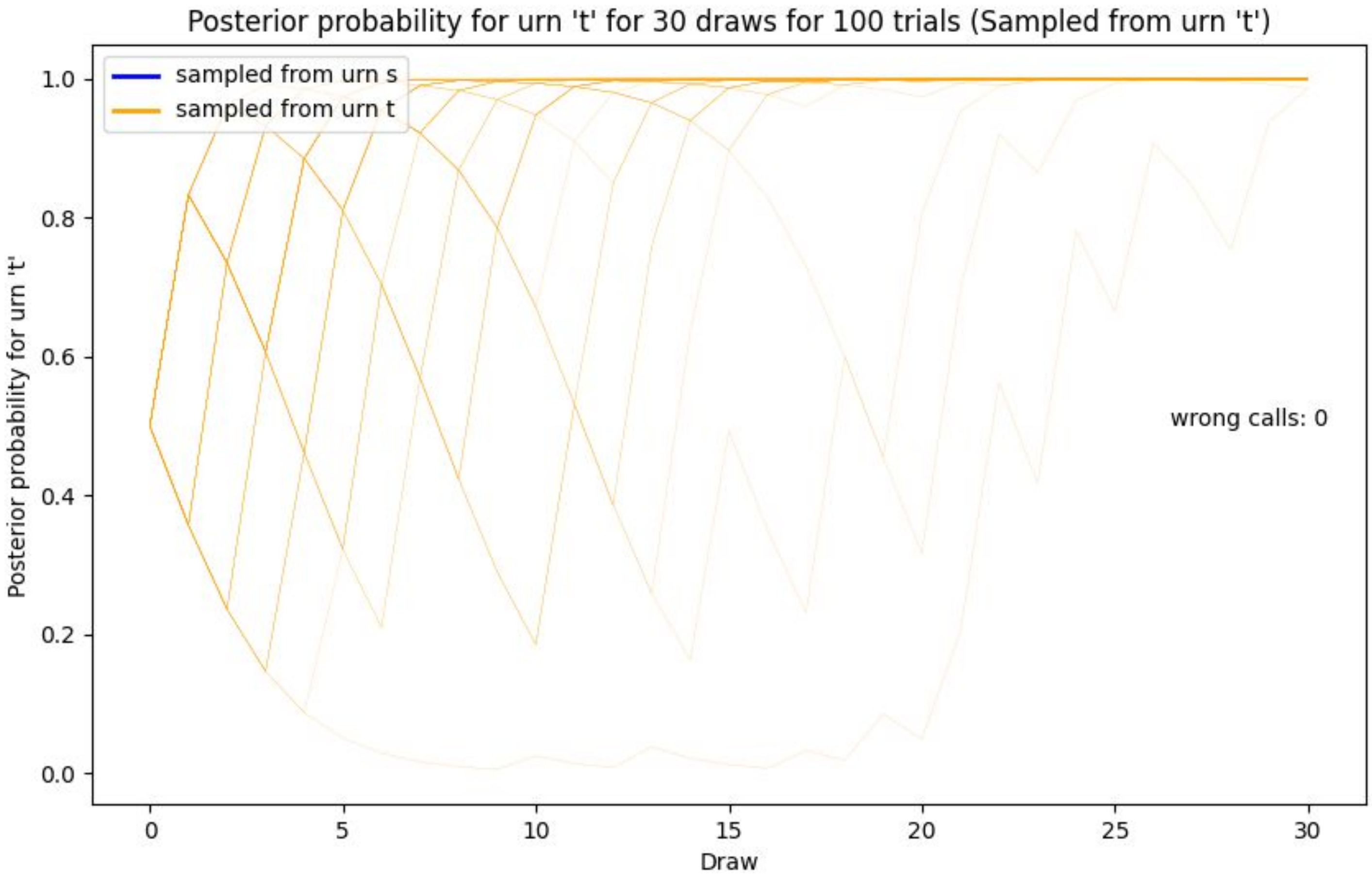


If 100 people picked urn ‘t’





Posterior probability for urn 't' for 30 draws for 100 trials (Sampled from urn 's')
sampled from urn s
sampled from urn t
wrong calls: 0
Posterior probability for urn 't'
0.0
0.2
0.4
0.6
0.8
0
5
10
15
20
25
30
Draw

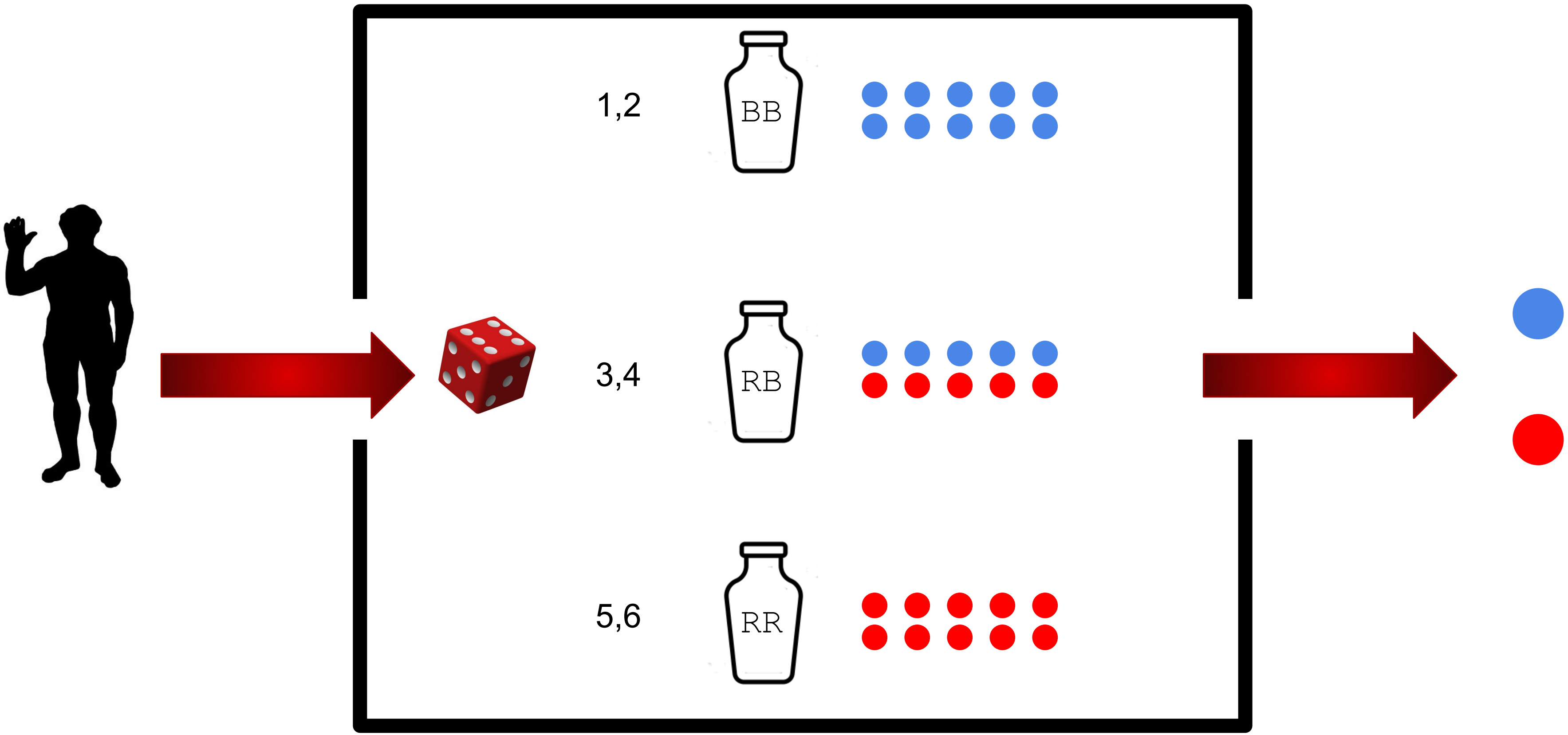


Posterior probability for urn 't' for 30 draws for 100 trials (Sampled from urn 't')
sampled from urn s
sampled from urn t
wrong calls: 0
Posterior probability for urn 't'
Draw
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30

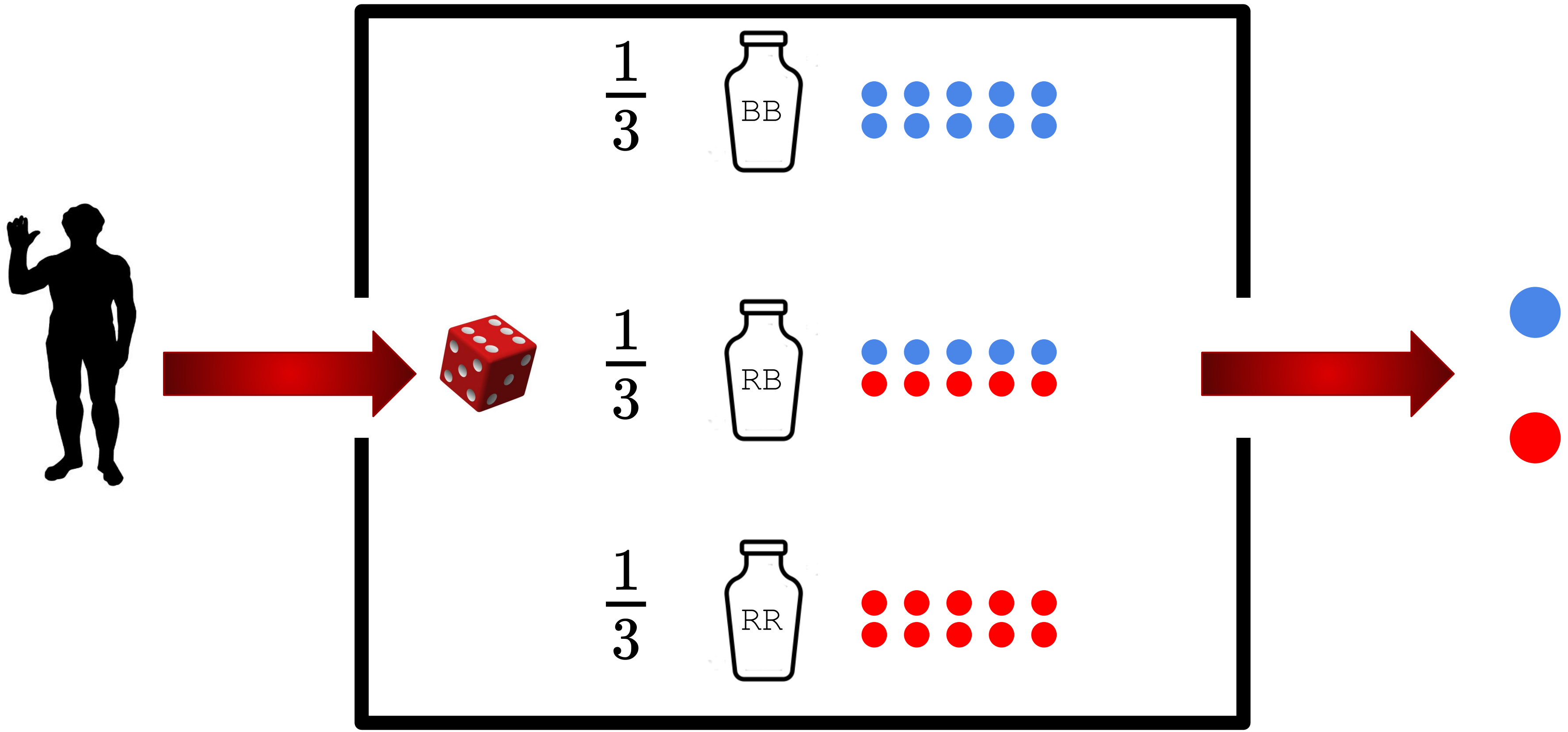
Key ideas

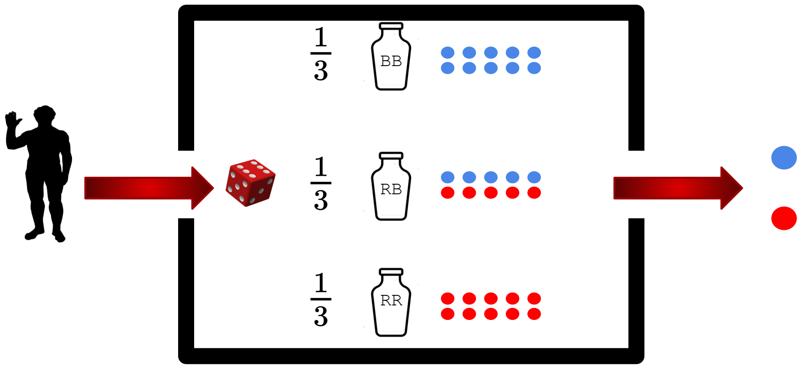
- **Additional independent observations** can give us more confidence in a model being the correct one
- Confidence is **never** absolute

A little twist



A little twist

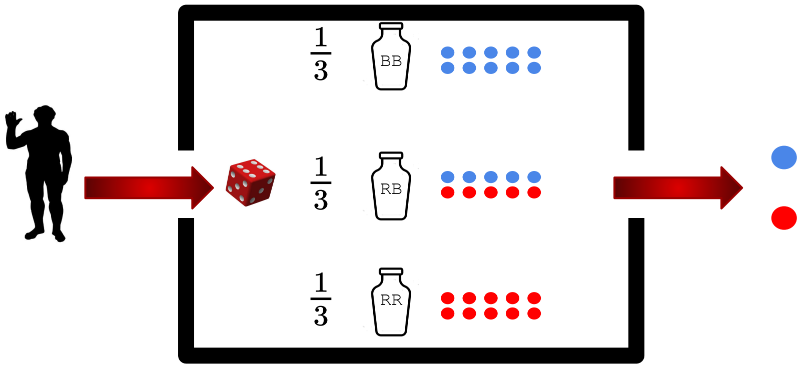




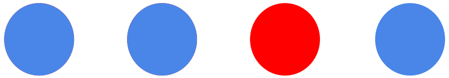
$$P[BB|D]$$

$$P[RB|D] \quad ?$$

$$P[RR|D]$$



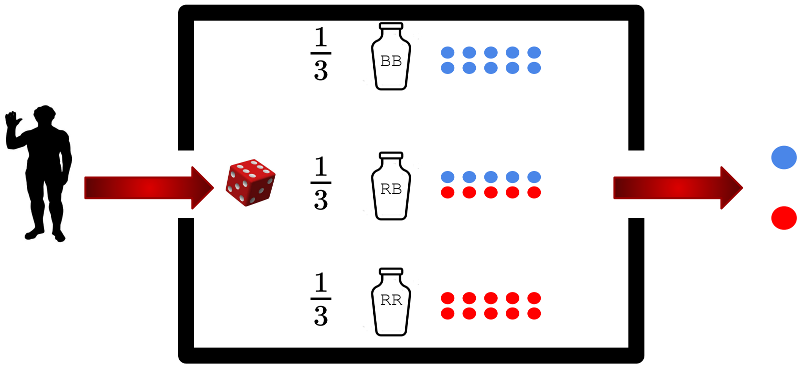
Observations 1:

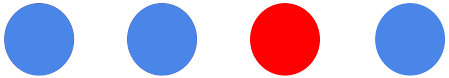


$$P[D|BB] = P['b'|BB] \times P['b'|BB] \times P['r'|BB] \times P['b'|BB]$$

$$P[D|BB] = 1 \times 1 \times 0 \times 1$$

$$P[D|BB] = 0$$

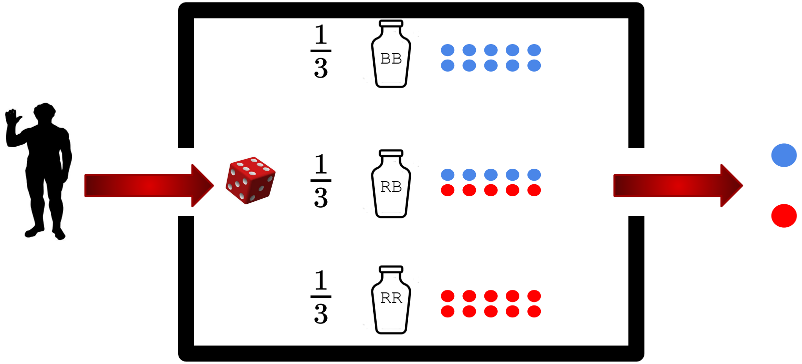


Observations 1: 

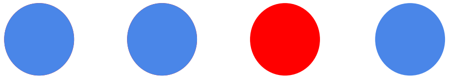
$$P[D|RB] = P['b'|RB] \times P['b'|RB] \times P['r'|RB] \times P['b'|RB]$$

$$P[D|RB] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$P[D|RB] = \frac{1}{16}$$



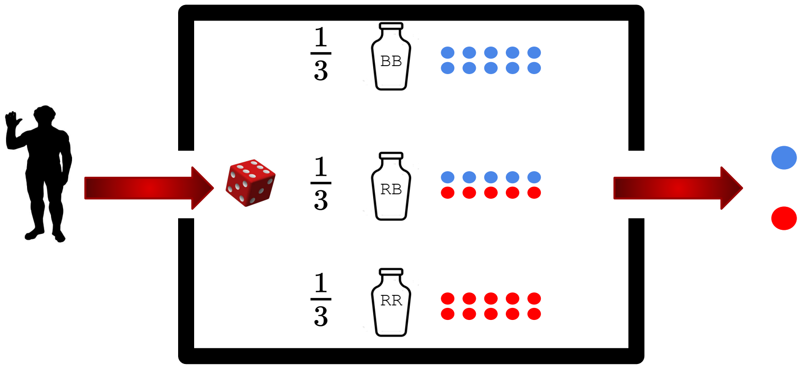
Observations 1:

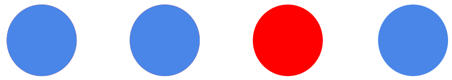


$$P[D|RR] = P['b'|RR] \times P['b'|RR] \times P['r'|RR] \times P['b'|RR]$$

$$P[D|RR] = 0 \times 0 \times 1 \times 0$$

$$P[D|RR] = 0$$



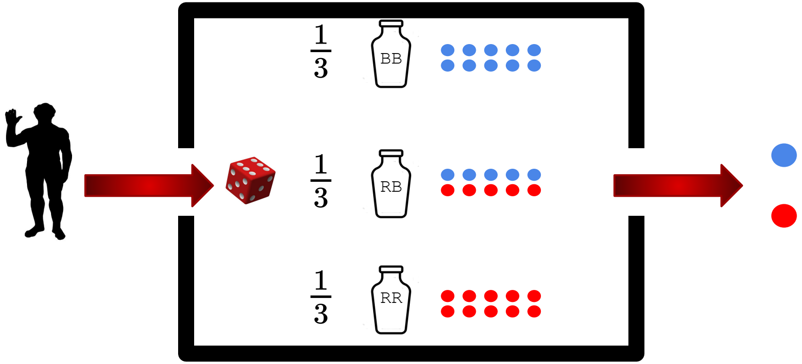
Observations 1: 

$$P[RB|D] = \frac{P[RB]P[D|RB]}{P[D]}$$

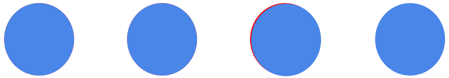
$$P[RB|D] = \frac{P[RB]P[D|RB]}{P[BB]P[D|BB]+P[RB]P[D|RB]+P[RR]P[D|RR]}$$

$$P[RB|D] = \frac{\frac{1}{3}\frac{1}{16}}{\frac{1}{3}0+\frac{1}{3}\frac{1}{16}+\frac{1}{3}0}$$

$$P[RB|D] = 1$$



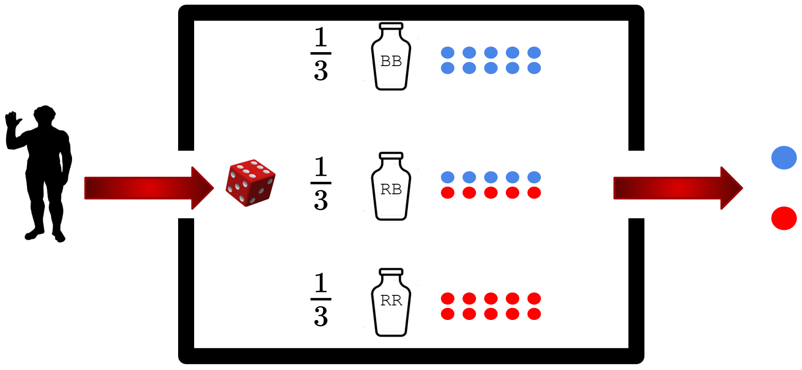
Observations 2:

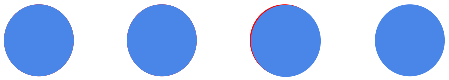


$$P[D|BB] = P['b'|BB] \times P['b'|BB] \times P['r'|BB] \times P['b'|BB]$$

$$P[D|BB] = 1 \times 1 \times 1 \times 1$$

$$P[D|BB] = 1$$

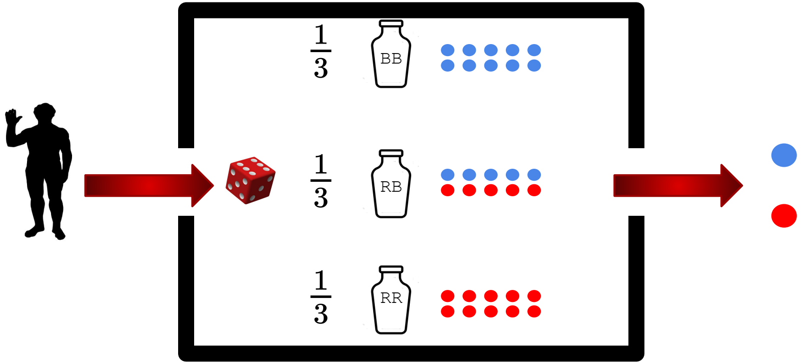


Observations 2: 

$$P[D|RB] = P['b'|RB] \times P['b'|RB] \times P['b'|RB] \times P['b'|RB]$$

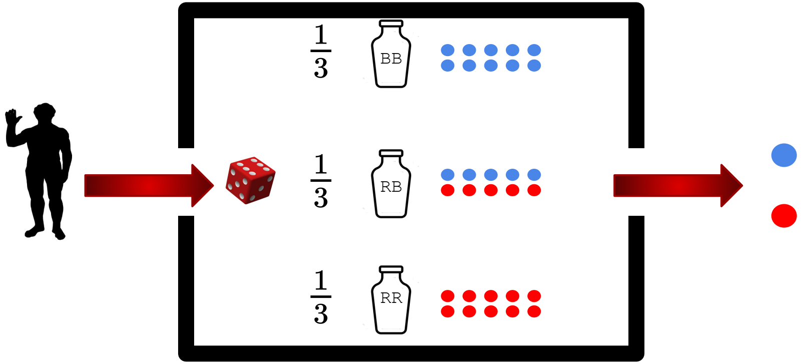
$$P[D|RB] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

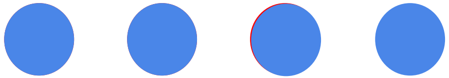
$$P[D|RB] = \frac{1}{16}$$



Observations 2:

$$P[D|RR] = 0$$

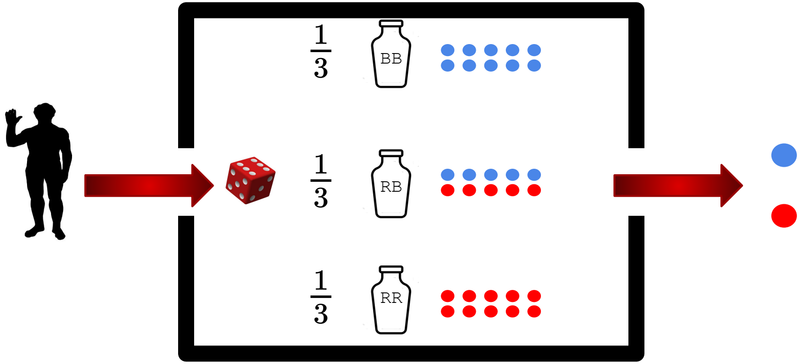


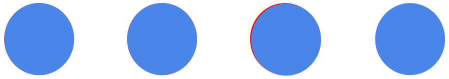
Observations 2: 

$$P[RB|D] = \frac{P[RB]P[D|RB]}{P[BB]P[D|BB]+P[RB]P[D|RB]+P[RR]P[D|RR]}$$

$$P[RB|D] = \frac{\frac{1}{3}\frac{1}{16}}{\frac{1}{3}1+\frac{1}{3}\frac{1}{16}+\frac{1}{3}0}$$

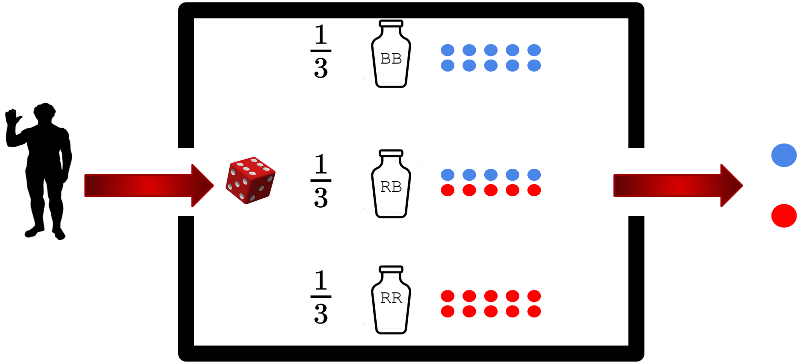
$$P[RB|D] = \frac{1}{17}$$



Observations 2: 

$$P[BB|D] = \frac{P[BB]P[D|RB]}{P[BB]P[D|BB]+P[RB]P[D|RB]+P[RR]P[D|RR]}$$

$$P[RB|D] = \frac{\frac{1}{3}1}{\frac{1}{3}1+\frac{1}{3}\,\frac{1}{16}+\frac{1}{3}0} = \frac{16}{17}$$



Observations 2:

$$P[RR|D] = 0$$

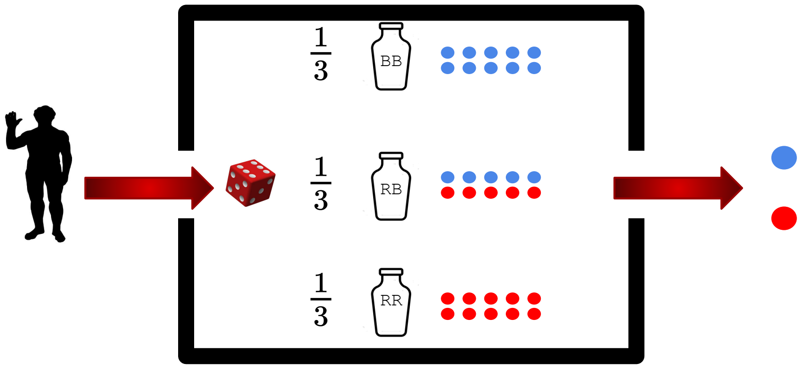
$$\frac{9}{10} = 0.9$$

it's red!

$$\frac{1}{10} = 0.1$$

it's blue!

source: DALL-E

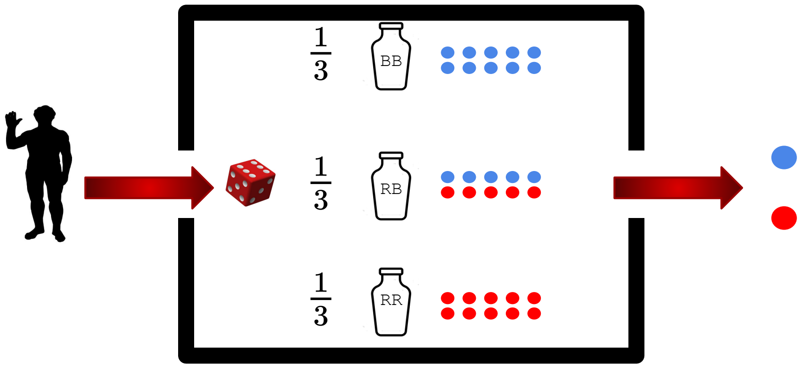


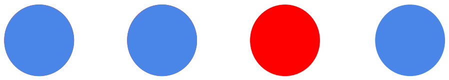
Observations 1:

$$P[D|BB] = P['b'|BB] \times P['b'|BB] \times P['r'|BB] \times P['b'|BB]$$

$$P[D|BB] = \frac{9}{10} \times \frac{9}{10} \times \frac{1}{10} \times \frac{9}{10}$$

$$P[D|BB] = \frac{729}{10000} = 0.0729$$



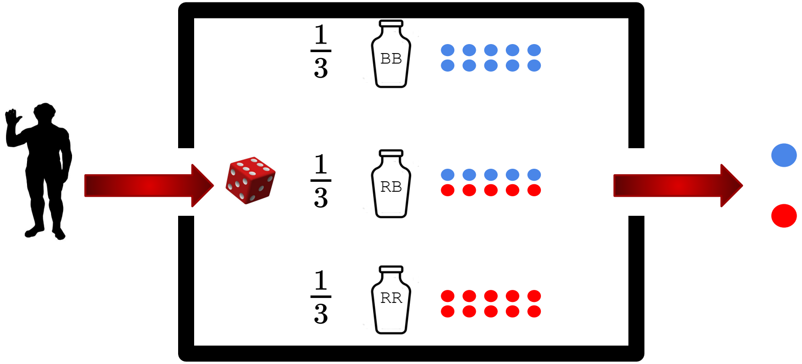
Observations 1: 

$$P[D|RB] = P['b'|RB] \times P['b'|RB] \times P['r'|RB] \times P['b'|RB]$$

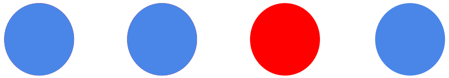
$$P['b'|RB] = P['B']P['b'|'B'] + P['R']P['b'|'R']$$

I sampled blue and called it correctly

I sampled red and called it blue by mistake



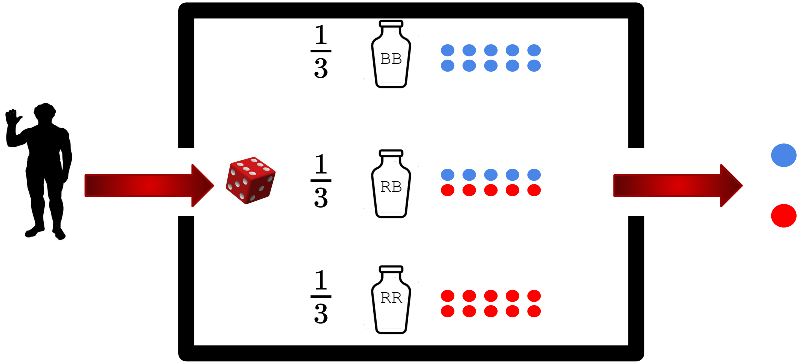
Observations 1:



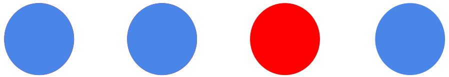
$$P[D|RB] = P['b'|RB] \times P['b'|RB] \times P['r'|RB] \times P['b'|RB]$$

$$P['b'|RB] = P['B']P['b'|'B'] + P['R']P['b'|'R']$$

$$P['b'|RB] = \frac{1}{2}\frac{9}{10} + \frac{1}{2}\frac{1}{10} = \frac{1}{2}$$



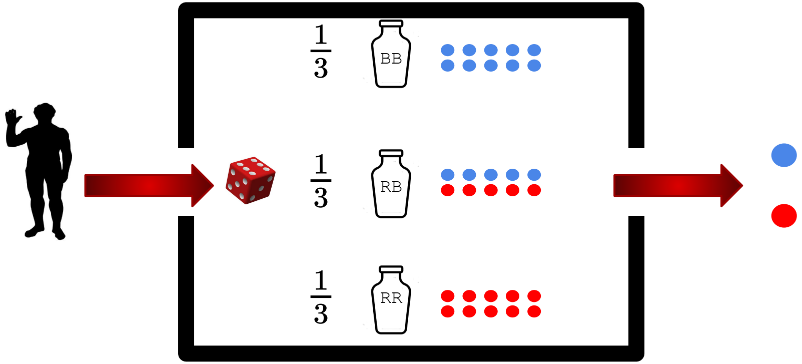
Observations 1:



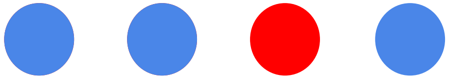
$$P[D|RB] = P['b'|RB] \times P['b'|RB] \times P['r'|RB] \times P['b'|RB]$$

$$P[D|RB] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$P[D|RB] = \frac{1}{16}$$

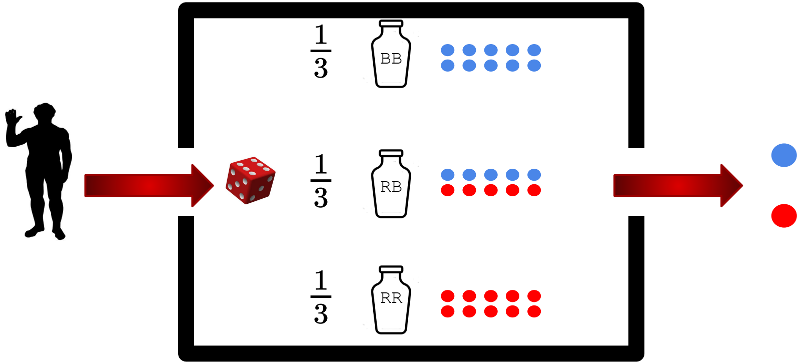


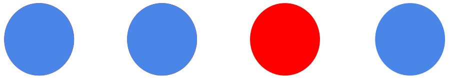
Observations 1:



$$P[D|RR] = P['b'|RR] \times P['b'|RR] \times P['r'|RR] \times P['b'|RR]$$

$$P['b'|RR] = \frac{1}{10} \times \frac{1}{10} \times \frac{9}{10} \times \frac{1}{10} = \frac{9}{10000}$$

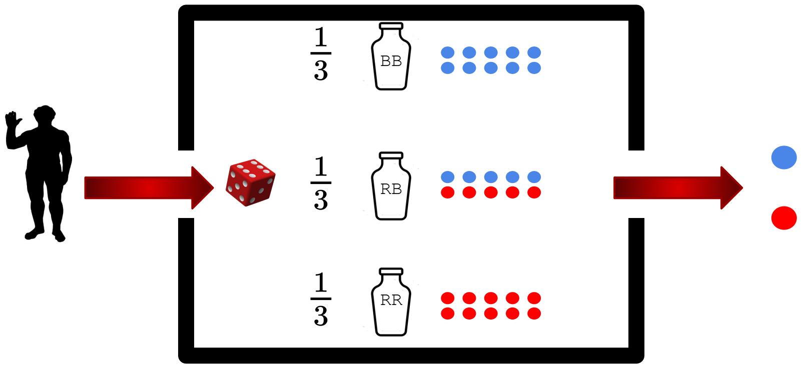


Observations 1: 

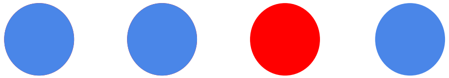
$$P[RB|D] = \frac{P[RB]P[D|RB]}{P[D]}$$

$$P[RB|D] = \frac{P[RB]P[D|RB]}{P[BB]P[D|BB]+P[RB]P[D|RB]+P[RR]P[D|RR]}$$

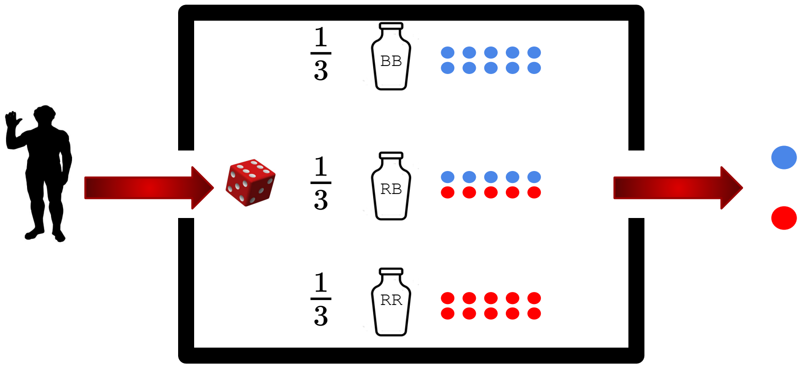
$$P[RB|D] = \frac{\frac{1}{3}\frac{1}{16}}{\frac{1}{3}\frac{729}{10000}+\frac{1}{3}\frac{1}{16}+\frac{1}{3}\frac{9}{10000}} = \frac{\frac{1}{16}}{\frac{1363}{10000}} \approx 0.458$$



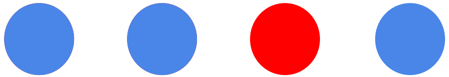
Observations 1:



$$P[BB|D] = \frac{\frac{1}{3}\frac{9}{10000}}{\frac{1}{3}\frac{729}{10000}+\frac{1}{3}\frac{1}{16}+\frac{1}{3}\frac{9}{10000}} = \frac{\frac{729}{10000}}{\frac{1363}{10000}} \approx 0.535$$



Observations 1:



$$P[RR|D] = \frac{\frac{1}{3}\frac{9}{10000}}{\frac{1}{3}\frac{729}{10000} + \frac{1}{3}\frac{1}{16} + \frac{1}{3}\frac{9}{10000}} = \frac{\frac{9}{10000}}{\frac{1363}{10000}} \approx 0.0066$$

Observations 1:

$$P[RB|D] \approx 0.535$$

$$P[BB|D] \approx 0.458$$

$$P[RR|D] \approx 0.0066$$

$$\frac{0.535}{0.458} \approx 1.168$$

Key ideas (2)

- When **errors in the observation exist**, we could not discard a particular model
- We care about the **ratio between posterior probabilities** as a measure of confidence